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Thom Polynomials in Symplectic Geometry

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In this note, we give topological invariants of $\Sigma_{1,\ldots,1}$-type singularities for isotropic mappings, namely, for Lagrange immersions with singularities. The simplest one of our formulae can be considered as the symplectic version of Rieman-Hurwitz formula.

§0 Introduction

Let L be an n-dimensionl C^∞ manifold, and (M,ω) a 2n-dimensional symplectic manifold (i.e. ω is a nondegenerate closed 2-form on M). A C^∞ mapping $f: L \to M$ is called *isotropic* if $f^*\omega$ is null on L. In particular, an isotropic immersion is usually called a *Lagrange immersion.*

The notion of Lagrange immersions plays an important role in classical mechanics, geometric optics, asymptotic analysis on solutions of nonlinear P.D.E ... etc, and there have been many works on the topology of Lagrange immersions from various viewpoints. On the other hand, for Lagrange immersions with singularities (i.e. non-immersive points), the local theory is recently developed (see Ishikawa [I], Givental' [G]), but few are results on their global topological aspects. As far as the author knows, there is only the work of Givental' concerning singular Lagrange surfaces in a 4-dimensional symplectic manifold (see [G]). Here we introduce the result of Givental' in a simple situation as follows.

Theorem (0.1) (Givental' [G]). *Let S be a closesd surface, $M = \mathbf{R}^4$ provided with the canonical symplectic structure, and $f: S \to \mathbf{R}^4$ an isotropic mapping which has the open Whitney umbrellas as its singularities (see Remark below). Then,*
(1) if S is orientable, the number of singular points of f taking accounts of signs is equal to zero (the sign of a singular point is defined by the local Maslov index at that point).
(2) If S is non-orientable, the number of singular points of f is equal to the Euler number of L modulo 2.

Remark (0.2) ([G]): (1) An open Whitney umbrella is described by the following local model: $f_{2,1} : \mathbf{R}^2, 0 \to \mathbf{R}^4, 0$ defined by $f_{2,1}(u,v) = (p_1, q_1, p_2, q_2) = (v^3/3, u, uv, v^2/2), \omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$. Note that the map f in the above (0.1) has isolated singular points. (2) We will review the definition of the local Maslov index in a general situation in §3.

In this paper, we generalize the above formulae (1) and (2) of Givental' from the viewpoint of Thom polynomials in the singularity theory (see Porteous [P]).

As another appraoch to the original work of Givental' [G], Ishikawa and the author obtained formulae of integrations some local invariants over a real singular surface in an almost complex manifold of dimension 4 ([I-O]).

Throughout this paper, all manifolds and maps are of class C^∞.

§1 Preliminary and the formulation of results

Let $f : L^n \to M^{2n}$ be isotropic, and assume that its differential df_x at each $x \epsilon L$ has kernel rank at most 1.

We use Thom's geometric descriptions of singularity sets of f, considering f simply as a C^∞ mapping : Set $\Sigma_1(f) = \{x \epsilon L | dim Ker(df)_x = 1\}$, and if $\Sigma_1(f)$ is a C^∞ submanifold of L, then set $\Sigma_{1,1}(f) := \Sigma_1(f|_{\Sigma_1(f)})$. We may continue to define $\Sigma_{1^k}(f)(= \Sigma_{1,\dots,1}(f)$ with 1 repeated k times) as $\Sigma_1(f|_{\Sigma_{1^{k-1}}(f)})$ inductively.

By using Ishikawa's theorem in [I], we can see that if f is in some open dense subset **G** of the space of all isotropic mappings from L to M with kernel rank at most 1, then each $\Sigma_{1^k}(f)(1 \le k \le [\frac{n}{2}])$ is well-defined, and the codimension of $\Sigma_{1^k}(f)$ in L is equal to 2k. **G** is also defined in terms of suitable transversal properties on intrinsic derivatives.

We let $P_k(f)$ (resp. $P'_k(f)$) denote the cohomology class dual to $[\Sigma_{1^k}(f)]$ in $H^{2k}(L, \mathbf{Z})$ (resp. $H^{2k}(L, \mathbf{Z}_2)$).

According to [W] every symplectic manifold (M, ω) admits an almost complex structure J such that $\omega(*, J*)$ is positive definite, and J is uniquely determined up to homotopy. Here we fix such a J, by which we will consider the tangent bundle TM as a complex vector bundle over M. Remark that Chern classes $c_i(TM)$ are independent of the choice of J.

We now state the main results in this paper.

Theorem (1.1). *Let $f : L^n \to M^{2n}$ be an isotropic mapping in* $\mathbf{G}$, *and let* $1 \leq k \leq [\frac{n}{2}]$. *Then, (1) if L is orientable, then $2c_i(f^*TM - TL_{\mathbf{C}}) = 0$ for $(i \geq 2)$, and $P_k(f) = f^*c_1(TM)^k$ modulo elements of order 2 , and*
(2) if L is non-orientable, then $P'_k(f) = \sum_{j=0}^{[\frac{k}{2}]} \sigma_j w_2^{k-j-1} w_{2j+2}$, where w_s means the s-th Stiefel-Whitney class of the difference bundle $f^*TM - TL \oplus TL$, and the number σ_j denotes $\binom{[\frac{k}{2}]}{j}$.

In the case of n=2, we have the following formulae, which are slightly different from Givental's (0.1) at the point that some characteristic numbers appear.

Corollary (1.2). *Let f be an isotropic mapping from a closed surface S to a 4-dimensional symplectic manifold M with open Whitney umbrellas as its singularities. Then,*
(1) if L is orientable, the number of singular points of f taking accounts of signs is equal to the Chern number $< f^*c_1(TM), [S] >$ (the sign is given by the local Maslov index at that point), and
(2) if L is non-orientable, the number of singular points (mod.2) is equal to the sum of $\chi(S)$(mod.2) and the Stiefel-Whitney number $< f^*w_2(TM), [S]_2 >$.

§2 Outline of the proof for Σ_1-type

Let $f : L^n \to M^{2n}$ be an isotropic mapping with kernel rank at most 1, and let TM' denote f^*TM.

First, we define the tautological ($\mathbf{R}$-linear) bundle isomrophism

$$\rho : Hom_{\mathbf{R}}(TL, TM') \to Hom_{\mathbf{C}}(TL_{\mathbf{C}}, TM')$$

by $\rho(h)(u,v) := h(u) + Jh(v)$ for $(u,v) \epsilon TL_{p_{\mathbf{C}}}$. If $h : TL_p \to TM'_p$ is an isotropic linear map, we can see that $\ker \rho(h) = (\ker h)_{\mathbf{C}} \subset TL_{p_{\mathbf{C}}}$. Thus $\Sigma_1(f)$ coincides with $(\rho \circ df)^{-1}\Sigma_1^{\mathbf{C}}$, where $\Sigma_r^{\mathbf{C}}$ denote the subbundle of $Hom_{\mathbf{C}}(TL_{\mathbf{C}}, TM')$ consisting of all $\mathbf{C}$-linear maps with kernel rank r.

Second, we assume that the induced section $\rho \circ df$ is transversal to $\Sigma_1^{\mathbf{C}}$ (see Remark (2.1)). It can be proved that this condition is independent of the choice of the almost copmlex structure J on TM and is a generic condition on the space of isotropic mappings with kernel rank at most 1.

Then we can apply Thom-Porteous' formula to $\rho \circ df$, and hence we obtain that $P_1(f) = f^*c_1(TM) - c_1(TL_{\mathbf{C}})$. In particular, in the case of n=2, this formula can be considered as the symplectic version of Rieman-Hurwitz formula.

For higher order singularities Σ_{1^k}, under the appropriate transversality conditions on the higher derivatives of f, we can apply the desingularization method due to Porteous [P] to obtain the formulae in Theorem (1.1). For the detail, see [O].

Remark (2.1): Unfortunately, it is non-sense to consider the similar approach to the case of kernel rank greater than 1. In fact, for any isotropic mapping g which has singularities of kernel rank r greater than 1, $\rho \circ dg$ is never transversal to $\Sigma_r^{\mathbf{C}}$. This comes from the fact that all isotropic linear mappings form an algebraic variety whose singular locus consists of isotropic linear mappings with kernel rank greater than 1 (see [I],[O]).

§3 The local Maslov class and the co-orientation of $\Sigma_1(f)$

In this section, we shall review the the local Maslov class, and complete the proof of (1) in Corollary (1.2).

First, let $f : L^n \to M^{2n}$ be isotropic simply, $\Sigma(f)$ the set of all singular points of f and $p \epsilon \Sigma(f)$. We assume L is oriented.

Recall the definition of the local Maslov class at p of f according to [I'] (or [G]): Choose a contractible neighborhood U around p and a trvialization of the symplectic bundle $\sigma : TM'|U \simeq U \times \mathbf{R}^{2n}$. Set $\Sigma = U \cap \Sigma(f)$, and then the restriction of f to $U - \Sigma$ is a Lagrange immersion. By using the trivialization σ, this restriction map induces the Gauss map ϕ_f from $U - \Sigma$ to the oriented Lagrange Grassmannian $\tilde{\Lambda}(n)$. Let μ be the Maslov class of $\tilde{\Lambda}(n)$, and then we define *the local Maslov class at p of f* , $m(f,p)$, as $\phi_f^* \mu \epsilon H^1(U - \Sigma; \mathbf{Z})$. $m(f,p)$ is independent of the choice of σ (see [I']).

Next, we assume that $p \epsilon L$ is a singular point at which df has kernel rank 1, and also that $\rho \circ df$ is transversal to $\Sigma_1^{\mathbf{C}}$ at p. Then, $\rho \circ df$ induces an orientation of the normal space to $\Sigma_1(f)$ at p, and hence it determines a generator of $H^1(U-\Sigma;\mathbf{Z})$ for sufficiently small U. It can be verified that

LEMMA. *This generator coincides with the local Maslov class $m(f,p)$.*

The proof of (1) of Corollary (1.2)

In the case of n=2, for an isolated singular point p of an isotropic mapping $f : S \to M$, we define *the local Maslov index of f at p* as $< \phi_f^* \mu, c > \epsilon \mathbf{Z}$, where c is the generator of $H_1(U-p;\mathbf{Z})$ compatible to the orientation of S.

We now assume that f has only open Whitney umbrellas as its singularities (i.e. $\rho \circ df$ is transversal to $\Sigma_1^{\mathbf{C}}$). Then, the above lemma yields that the sum of the local Maslov indices is equal to the transversal index of $\rho \circ df$ and $\Sigma_1^{\mathbf{C}}$, and hence we have (1) of Corollary (1.2) by the formula of $P_1(f)$. This completes the proof.

§3 Application to the ray system

Finally, we introduce a simple application of our formulae to the generalized Cauchy problem for an Hamilton-Jacobi equation defined in a symplectic manifold (M,ω) of dimension 2n. Our framework described below is based on [A],[A-M] and [G'] (i.e. the ray system in [A] and [G']).

Let E be a hypersurface in M with the normal bundle ν, ξ the characteristic line field on E (i.e. the skeworthogonal line bundle $TE^{\perp}$) and $F : M \times \mathbf{R} \to M$ the flow of ξ. Note that ν is canonicaly isomorphic to ξ.

Let N be an (n-1)-dimensional compact isotropic submanifold of M such that $N \subset E$. It is well known that if N is transverse to ξ, then $F(N \times \mathbf{R})$ is a Lagrange submanifold contained in E, which can be considered as *the geometric solution of the equation E with the initial data N*. If N is tangent to ξ at some points, $F(N \times \mathbf{R})$ becomes a Lagrange manifold with singularities, whose singularities appear along the union of integral curves through the non-transeversal points of N to ξ. A point x of N is called *tangential to ξ of order k* if the integral curve through x is tagent to N at x of order k. Let $S_k(\xi)$ be the set of the k-th order tangential points in N. For the initial data N in general position, $S_k(\xi)$

is a closed C^∞-submanifold of N with codimension 2k.

Now our interest is to find topological obstructions of the existence of $S_k(\xi)$. Applying above Theorem to this situation, for instance, we have

Corollary (1.3). *The Poincare dual to the homology class* $[S_1(\xi)]$ *of* N *is equal to* $c_1(i^*\tau_M - \nu_{\mathbf{C}} - TL_{\mathbf{C}})$.

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